Asymptotic constructions and invariants of graded linear series

Shin-Yao Jow

Department of Mathematics
National Tsing Hua University

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Notation

- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathbf{K}$ = alg. closed field.
- X = projective variety over $\mathbf{K}$.
- L = line bundle on X.

Linear series (linear systems)

- A *linear series (linear system)* associated to L on X is a vector subspace $V \subseteq H^0(X, L)$. Ex: $V = H^0(X, L)$ is called the *complete linear series*.
- A (nonzero) linear series $V \subseteq H^0(X, L)$ defines a rational map $\phi\colon X \dashrightarrow \mathbb{P}(V)$. (In terms of a basis $\{s_0, \ldots, s_n\}$ of V, $\phi\colon X \dashrightarrow \mathbb{P}(V) \cong \mathbb{P}^n$ is given by $p \mapsto [s_0(p) : \cdots : s_n(p)]$.)

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Graded linear series

- A *graded linear series* associated to L on X is a collection $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ of vector subspaces $V_m \subseteq H^0(X, L^m)$ such that $V_0 = \mathbf{K}$ and $V_k \cdot V_\ell \subseteq V_{k+\ell}$ for all $k, \ell \in \mathbb{N}$.
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Asymptotic Iitaka map of graded linear series

$V_\bullet = \{V_m\}_{m\in\mathbb{N}}$, $V_m \subseteq H^0(X, L^m)$: graded linear series.

- Each nonzero V_m defines a rational map
$$\phi_m\colon X \dashrightarrow Y_m \subseteq \mathbb{P}(V_m), \quad Y_m = \overline{\phi_m(X)}.$$
- $\mathbf{N}(V_\bullet) = \{m \in \mathbb{N} \mid V_m \neq 0\}$: semigroup, assume $\neq \{0\}$.

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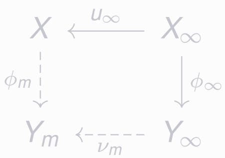
As $m \to \infty$, the maps ϕ_m stabilize birationally: There exist projective varieties X_∞, Y_∞, and for each $m \in \mathbf{N}(V_\bullet)$ a commutative diagram



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As $m \to \infty$, $\dim_{\mathbf{K}} V_m \approx m^\kappa$: The limit

$$\operatorname{vol}_\kappa(V_\bullet) = \lim_{m \in \mathbf{N}(V_\bullet)} \frac{\dim_{\mathbf{K}} V_m}{m^\kappa/\kappa!}$$

exists, and $0 < \operatorname{vol}_\kappa(V_\bullet) < \infty$. We call it the κ-volume of $V_\bullet$.

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Moving intersection number

- $Z \subseteq X$ closed subvariety, $\dim Z = k$.
- $V \subseteq H^0(X, L)$ nonzero subspace.
- $\mathrm{Bs}(V) =$ base locus of V.

The *moving intersection number* of V with Z, denoted by $(V^k \cdot Z)_{\mathrm{mov}}$, is defined by choosing k general divisors $D_1, \ldots, D_k \in |V|$ and putting

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Theorem

Let $Z \subseteq X$ be a general κ-dim closed subvariety. Then the limit

$$(V_\bullet^\kappa \cdot Z)_{\mathrm{mov}} = \lim_{m\in\mathbf{N}(V_\bullet)} \frac{(V_m^\kappa \cdot Z)_{\mathrm{mov}}}{m^\kappa}$$

exists, and $0 < (V_\bullet^\kappa \cdot Z)_{\mathrm{mov}} < \infty$. Moreover,

$$(V_\bullet^\kappa \cdot Z)_{\mathrm{mov}} = \delta(V_\bullet|_Z)\,\mathrm{vol}_\kappa(V_\bullet) = \deg\left(\phi_m|_Z \colon Z \dashrightarrow \phi_m(Z)\right)\mathrm{vol}_\kappa(V_\bullet)$$

for large $m \in \mathbf{N}(V_\bullet)$. In particular, if $\kappa = n = \dim X$, then

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Asymptotic moving intersection number and κ-volume

$V_\bullet = \{V_m\}_{m\in\mathbb{N}}$, $V_m \subseteq H^0(X, L^m)$: graded linear series.

- Each nonzero V_m defines a rational map
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Theorem

Let $Z \subseteq X$ be a general κ-dim closed subvariety. Then the limit

$$(V_\bullet^\kappa \cdot Z)_{\mathrm{mov}} = \lim_{m\in\mathbf{N}(V_\bullet)} \frac{(V_m^\kappa \cdot Z)_{\mathrm{mov}}}{m^\kappa}$$

exists, and $0 < (V_\bullet^\kappa \cdot Z)_{\mathrm{mov}} < \infty$. *Moreover,*

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